MACFARLANE'S ALGEBRA OF PHYSICS.*

By PROF. E. W. HYDE, Cincinnati, Ohio.

Before entering upon a careful consideration of Professor Macfarlane's paper, I wish to express my hearty appreciation of the skill and ability shown by him in the development of a method of multiplication so interesting from an algebraic point of view, as well as in his analysis of the fundamental principles of the subject. The weighty objections to the system of Quaternions as developed by Hamilton are well and strongly stated, and the Author seeks to develop a system which shall possess what he regards as the advantages of Hamilton's without its drawbacks. The recognition and separate treatment of products and quotients of vectors as things essentially different is certainly a great improvement over the quaternion treatment.

The opening sentences of the paper appear to indicate that the Author has been partly moved by a desire to vindicate the work of Hamilton and Tait against those who have asserted the superiority of Grassmann's system.

The Author claims to have devised "*a more complete algebra which unifies Quaternions, Grassmann's Method, and Determinants, and applies to physical quantities in space.*" It is proposed to examine this claim somewhat in detail. As to the last specification, there is no doubt of the *fact;* the only question is as to relative superiority of the different systems.

The defect asserted with regard to Grassmann's system is that in it we do not have a *complete* distributive product, because the product of identical factors is zero.† In ordinary algebra we have $(a + b + c)(a' + b' + c') = aa' + ab' + ac' + ba' + bb' + bc' + ca' + cb' + cc'$, and we should have a similar result in directional algebra.

But do we not have that very thing? Let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be any six vectors; then by the *Ausdehnungslehre* $(\alpha + \beta + \gamma)(\alpha' + \beta' + \gamma') = \alpha\alpha' + \alpha\beta' + \alpha\gamma' + \beta\alpha' + \beta\beta' + \beta\gamma' + \gamma\alpha' + \gamma\beta' + \gamma\gamma'$, precisely as in scalar algebra, and, in general, none of the partial products is zero. Now suppose the quantities $\alpha, \ldots, \alpha'$, to vary continuously according to definite laws; then they will, in general, pass one or more at a time, through the value zero. At the instant of such passage of any one of the quantities three of the partial products disappear. Does that vitiate the multiplication? Similarly let $\alpha, \ldots, \gamma'$ vary. Now, 1°, their

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† See bottom of p. 77.

tensors can change while their directions remain constant, giving precisely the same results as with $a, \ldots, c'$, and, 2°, their *directions* can change while the tensors are constant. In this case among the infinite number of sets of simultaneous values a certain comparatively small number will cause one or more partial products to vanish because of parallelism. Why should this form a valid objection to this kind of multiplication? 3°. Both lengths and directions of the vectors may change, which simply combines the two previous results. It appears then that we *do* have in Grassmann's system a perfectly general and complete distributive product.

At the bottom of p. 77 it is asserted that "as a consequence of not treating together the two complementary parts of the product of two vectors, Grassmann and his followers have *restricted their attention to associative products*" (Italics mine). The fact is that Grassmann's multiplication is only *partially* associative: what he calls progressive and regressive products are such, but mixed products are *not*. Thus, if p be a point, and L_1 and L_2 be point-vectors (fixed in position), then $pL_1 \,.\, L_2$ is not the same as $p \,.\, L_1L_2$, in either two or three dimensional space. In the same place it is said "that is a very arbitrary principle which holds that all the terms into which two similar directions enter must vanish." It is difficult to see in this anything *more* arbitrary than in the assumption that the product of a vector by itself should be the square of its tensor. Grassmann's result is obtained by the most natural and reasonable extension of the idea of the square of a scalar, and represents perfectly the fact that if a cell with four equal sides be closed up by making two opposite angles zero and the other two each 180°, the enclosed area vanishes.

Towards the bottom of p. 71 occurs the following remark: "This kind of product, in which the factors are vectors, has in recent times been generally neglected." A statement is then quoted from Clifford to the effect that in any product all factors but the last are *operators*. Now the combinatory products of Grassmann are essentially of the kind spoken of by the Author as neglected, $\alpha\beta$ being the product of the two independent vectors α and β, and not at all the operation α on β, and so for all products.

Let us now consider the claim of the Author to have devised "a more complete algebra which unifies Quaternions, Grassmann's method, and Determinants." We have already seen that Grassmann's products have the complete distributive quality as much as Professor Macfarlane's; we will now look at the two systems from a geometric standpoint. To be sure, our Author styles his work the "Algebra of Physics;" but what do we treat in Physics by means of our equations but the geometric relations of the quantities involved? Now, there enter into Grassmann's system *all* the geometric quantities that exist, as the point, fixed line, fixed plane, line-direction, and plane-direction

for three-dimensional space, and similarly for space of four, five, or n dimensions; and we have a complete system of geometric multiplication involving all of these, which adapts the method wonderfully to the purposes of geometry. In the proposed system we have only the line direction as in Quaternions, which is to be supplemented by the notation $A\,.\,F$ for a fixed line, A and F being simply directed lines, or vectors. Now, of course, we can if we choose regard this as meaning that A is drawn out from some fixed point, and then F through the end of it; but is it quite logical to combine two things, of which neither is fixed in position, and call the combination a *fixed* vector? In the *Ausdehnungslehre*, on the contrary, p is a definite point whose multiplication into the definite direction ε make the definite line $p\varepsilon$ through p in the direction ε; and this line we may, and do constantly, represent by a single letter as L, using either notation as may be convenient. The expression $A\,.\,F$ is thus used to designate the same thing as $p\varepsilon$, but can p therefore be properly said to be "equivalent" to A, as stated on p. 80? It is equivalent only in *use*, not in *meaning*; in the one case one quantity having only length and direction is arbitrarily used to fix in *position* another quantity having the same qualities, while in the other the idea of position in p is combined with those of direction and length in ε. It is difficult to see how such a notation could be used with any facility in treating of the screws, and wrenches, and twists of Professor Ball; while Grassmann's system works to perfection in this field.

Now, in view of these facts, can the proposed system be regarded as more complete than Grassmann's, even with reference only to three-dimensional space? What is done, as regards the product of two vectors, is simply to take as the, so-called, *complete* product the sum of Grassmann's *outer* and *inner* products, a complex expression, since one of these quantities is scalar and the other directed. The complexity is not practically helped by regarding the scalar part as having an indefinite axis, as is done by Professor Macfarlane, though it may be a relief to one's intellectual scruples. In the case of three or more vectors the results are far more complex, and seem to me to be of little utility. I am obliged to confess that the Author's arguments fail to convince me of the necessity or desirability of using this "complete" product. He says on p. 76, "The works of Hamilton and Tait make it abundantly evident that the quaternion idea is essential to the algebraic treatment of Spherical Trigonometry and of rotation." It is certainly true that the use of the "quaternion idea," i. e. the versor, or ratio of two vectors, simply as a versor, or turner, is convenient in treating rotations; but this can be done equally well with Grassmann' system as with Hamilton's or with Professor Macfarlane's, and does not at all involve the necessity of making the product of two or more vectors complex. Further, all the fundamental formulæ of Spherical Trigo-

nometry are derived with the utmost ease by Grassmann's methods without using the versor, as well as those of Plane Trigonometry except De Moivre's; though there is no particular practical advantage in so doing, because students rarely, if ever, study directional algebra till they have mastered Trigonometry.

Another word with reference to point analysis. Professor Macfarlane uses the equation $p_2 = p_1 + \varepsilon$ to show that Grassmann's system involves heterogeneous equations, and argues that, if ε is a vector, p_1 and p_2 must be vectors also, so that the point-analysis becomes really a vector-analysis. (See p. 79.) The fact is, the case is just the other way, and ε is a *point*, viz. a point of zero weight at ∞, and is *always* to be so regarded when it appears in an equation with points.

Now what do we know about such a zero point at ∞? Simply its *direction*, and the fact that it changes the position of the mean point of any system of points to which it may be added by a certain definite distance in its own direction, on account of its infinite *arm*. This distance in a definite direction is the essence of a vector, which assigns the reason for frequently calling p_∞ a vector.

We have a precisely similar thing in forces and couples. A couple is simply the sum of two forces equal in magnitude, parallel, and opposite; it must thus be of the same kind, and is, in fact, a zero force at ∞, and yet it appears as of two dimensions when regarded simply as a couple.

Arguing in the same line (p. 79) the Author says, "From the physical point of view it is more correct to treat of a mass-vector than of a point having weight; for the differential coefficient with respect to time of a mass-vector is the momentum, which is itself a mass-vector. If the latter is of one dimension in length so is the former. The product of a point and a mass is not a physical idea." But the differential coefficient with respect to time of a weighted point is *likewise* the momentum, the differential of a point being always a vector, and the argument falls to the ground. As to the product of a point and a mass not being a physical idea; we certainly deal in physics with masses situated at points, and why should not the combination of the two be regarded as a product as much as the combination of a length and a direction?

Towards the bottom of p. 87 the Author makes the following statement: "By the complementary vector (Fig. 12) of A with respect to B, Grassmann means the vector which has the same magnitude as A and is drawn perpendicular to A in the plane of A and B." This is certainly a mistake. There is no such thing as a line-vector complementary to a line-vector in Grassmann's system in space of higher dimensions than *two*. In three-dimensional space, which is here under consideration, the complement of a vector is a perpendic-

ular *plane*-vector whose tensor (area) is equal numerically to the tensor (length) of the vector, which is a very different conception. The vector is of *one* dimension, its complement is of *two*. This brings out again the fact that *only* line-vectors enter into Hamilton's or Macfarlane's systems, whereas in four-dimensional space we should have solid-vectors, as well as plane-vectors, and so on for spaces of higher orders. To be consistent with his own notation Professor Macfarlane should write $\overline{ij} = k$ instead of $ij = k$, etc., thus avoiding an apparent heterogeneity.

In view of what has been adduced it appears to me that Professor Macfarlane's claims for his method can not be regarded as valid. He has shown conclusively, in my judgment, that the combination of the vector and the versor in one and the same symbol, as was done by Hamilton, is neither necessary nor desirable, and has worked out a consistent system without that combination, treating separately product and ratio (quaternion) quantities, which may be regarded as an improvement on Hamilton's method ; but, in the opinion of the writer of this article, the methods of the *Ausdehnungslehre* so far excel those of Quaternions in completeness, generality, simplicity, and ease of interpretation and application, especially in geometry, but also in the realm of physics, that no idea of *unifying* the two can possibly be entertained.